III Geometric group theory – Example Sheet 1

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The notation $[x,y]$ denotes the *commutator* $xyx^{-1}y^{-1}$.

1. Let m,n be non-zero integers. The (m,n)-*Baumslag–Solitar group* $BS(m,n)$ has presentation $\langle a,b \mid ba^m b^{-1} a^{-n}\rangle$. Identify the presentation complexes of $BS(1,1)$ and $BS(1,-1)$ as well-known spaces.

2. Consider $G_n = \langle x,y,z \mid x^2 = y^3 = z\,,\ (xy)^{6n+5} = z^{5n+4})\rangle$ for $n \geq 0$, the fundamental groups of Dehn's homology spheres discussed in lectures.
   (a) Show that z is central in G_n, for all n.
   (b) Show that G_0 surjects the alternating group A_5.

3. Consider $BS(1,2) = \langle a,b \mid bab^{-1}a^{-2}\rangle$. For each n, exhibit a van Kampen diagram for the word
$$w_n = b^n a b^{-n} a b^n a^{-1} b^{-n} a^{-1}\ .$$
Comment on the areas of these van Kampen diagrams.

4. Consider the presentation $\mathbb{Z}^2 \cong \langle a,b \mid [a,b]\rangle$, and the word $w_n = [a^n, b^n]$ for $n \geq 1$. Show that the area of w_n is n^2. [*Hint: The presentation complex for* $\mathbb{Z}^2$ *induces a cell structure on the universal cover* $\widetilde{X} \cong \mathbb{R}^2$ *of the presentation complex. Consider the cellular homology of* $\widetilde{X}$.]

5. The subgroup of the isometry group of $\mathbb{R}$ that preserves the integers $\mathbb{Z}$ setwise is called the *infinite dihedral group* and is denoted by D_∞. Let r denote reflection in 0, let s denote reflection in 1/2, and let t denote translation by 1.
   (a) Show that $A = \{r,s\}$ and $B = \{r,t\}$ both generate D_∞.
   (b) By considering a point $x \in \mathbb{R}$ with trivial stabiliser, draw Cayley graphs for D_∞ with respect to both A and B.

6. Let $G = \langle A \mid R\rangle$, with A a finite generating set. For any $n \geq 0$ let B_n denote the closed ball around 1 in $\mathrm{Cay}_A(G)$ of radius n. Prove that the word problem is solvable in $\langle A \mid R\rangle$ if and only if there is an algorithm that takes as input an integer n and outputs the finite graph B_n.

7. Consider the presentation complex X of a group $G = \langle A \mid R\rangle$. Let Y be the covering space of the 1-skeleton $X_{(1)}$ that corresponds to $\langle\langle R\rangle\rangle$ under the Galois correspondence for covering spaces. Prove that Y is isomorphic to $\mathrm{Cay}_A(G)$.

8. Let F_n denote the free group of rank n, i.e. $F(\{a_1,\ldots,a_n\})$. Prove that F_2 contains subgroups isomorphic to F_n for all $n \geq 2$. Does it contain free subgroups of infinite rank?

9. (a) Let Y be a wedge of 4 circles. Draw the covering space Y' corresponding to the kernel of the homomorphism $\pi_1(Y) \to \mathbb{Z}/2\mathbb{Z}$ that sends every generator to 1. Write down an explicit injective homomorphism $F_7 \to F_4$. What is the index of the image of F_7 in F_4?
   (b) Let Σ_g denote the surface of genus g. Consider the presentation
$$\pi_1(\Sigma_2) \cong \langle a_1, a_2, b_1, b_2 \mid [a_1,b_1][a_2,b_2]\rangle\,,$$
and show that there is a homomorphism $f : \pi_1(\Sigma_2) \to \mathbb{Z}/2\mathbb{Z}$ with $f(a_i) = f(b_i) = 1$. Let X' be the covering space of Σ_2 corresponding to $\ker f$. Draw a schematic picture of a natural cell structure on X'. Prove that X' is homeomorphic to Σ_3 [*Hint: consider Euler characteristic*]. Write down a presentation of $\pi_1(\Sigma_3)$ with one relation, and an explicit injective homomorphism $\pi_1(\Sigma_3) \to \pi_1(\Sigma_2)$.

10. Let X be a topological space. Assuming that X is compact, locally compact, and has a universal cover, show that $\pi_1(X)$ is finitely generated.

11. Prove that quasi-isometry defines an equivalence relation on metric spaces.

12. Show that $d(x,y) = \sqrt{|x-y|}$ defines a metric on $\mathbb{R}$. Deduce that the hypothesis that X is geodesic cannot be dropped from the Schwarz–Milnor lemma.